Cayley-Hamilton theorem

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Let

$$\Delta(\lambda) = \det(\lambda I - A) = \lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_{n-1}\lambda + \alpha_n \tag{1}$$

be the characteristic polynomial of A. Then

$$\Delta(A) = A^n + \alpha_1 A^{n-1} + \cdots + \alpha_{n-1} A + \alpha_n I = 0 \tag{2}$$

Remark. Equation (2) suggests that a matrix satisfies its own characteristic polynomial. Further, it implies that A^n can be written as a linear combination of $\{I, A, \ldots, A^{n-1}\}$. And A^{n+1} can be written as a linear combination of $\{A, A^2, \ldots, A^n\}$, which, in turn, can be written as a linear combination of $\{I, A, \ldots, A^{n-1}\}$. Proceeding forward, we conclude that, for any polynomial $f(\lambda)$, no matter how large its degree is, $f(A)$ can always be expressed as

$$f(A) = \beta_0 I + \beta_1 A + \cdots + \beta_{n-1} A^{n-1} \tag{3}$$

In fact, for any function $f(\lambda)$, not necessarily a polynomial, we can still express $f(\lambda)$ in the form of (3).

Remark. This note is borrowed from the third edition of "Linear System Theory and Design" by Chi-Tsong Chen.